By the rank condition (42) and the Implicit Function Theorem (Theorem 15.7), there exist e coordinates $x_{j_1}, \ldots, x_{j_e}$ such that we can consider system (47) as implicitly defining $x_{j_1}, \ldots, x_{j_e}$ in terms of the rest of the x_i's and all the b_j's. In this latter set of exogenous variables, hold $b_2, \ldots, b_e$ constant, hold the exogenous x_j's constant, and let b_1 decrease linearly: $t \mapsto b_1 - t$ for $t \geq 0$. By the Implicit Function Theorem, as the exogenous variable b_1 varies, we can still solve system (47) for $x_{j_1}, \ldots, x_{j_e}$. This means, in particular, that there is a C^1 curve $\mathbf{x}(t)$ defined for $t \in [0, \varepsilon)$, such that $\mathbf{x}(0) = \mathbf{x}^*$ and, for all $t \in [0, \varepsilon)$,

$$g_1(\mathbf{x}(t)) = b_1 - t \quad \text{and} \quad g_j(\mathbf{x}(t)) = b_j \quad \text{for} \quad j = 2, \ldots, e. \tag{48}$$

Let $\mathbf{v} = \mathbf{x}'(0)$. Applying the Chain Rule to (48), we conclude that

$$Dg_1(\mathbf{x}^*)\mathbf{v} = -1, \quad Dg_j(\mathbf{x}^*)\mathbf{v} = 0 \quad \text{for} \quad j = 2, \ldots, e. \tag{49}$$

Since $\mathbf{x}(t)$ lies in the constraint set for all t and $\mathbf{x}^*$ maximizes f in the constraint set, f must be nonincreasing along $\mathbf{x}(t)$. Therefore,

$$\frac{d}{dt} f(\mathbf{x}(t))\bigg|_{t=0} = Df(\mathbf{x}^*)\mathbf{v} \leq 0.$$

Let $D_{\mathbf{x}}L(\mathbf{x}^*)$ denote the derivative of the Lagrangian (45) with respect to $\mathbf{x}$. By our first order conditions (46) and by (49),

$$\begin{aligned} \mathbf{0} &= D_{\mathbf{x}}L(\mathbf{x}^*)\mathbf{v} \\ &= Df(\mathbf{x}^*)\mathbf{v} - \sum_i \lambda_i Dg_i(\mathbf{x}^*)\mathbf{v} \\ &= Df(\mathbf{x}^*)\mathbf{v} - \lambda_1 Dg_1(\mathbf{x}^*)\mathbf{v} \\ &= Df(\mathbf{x}^*)\mathbf{v} + \lambda_1. \end{aligned}$$

Since $Df(\mathbf{x}^*)\mathbf{v} \leq 0$, we conclude that $\lambda_1 \geq 0$. A similar argument shows that $\lambda_j \geq 0$ for $j = 1, \ldots, e$. This finishes the proof of Theorem 18.4.

The argument just presented works equally well when the problem contains both inequality constraints and equality constraints, provided that NDCQ is valid at $\mathbf{x}^*$ for the combined equality and *binding* inequality constraints.

EXERCISES

19.24 Write out the proof that $\lambda_2 \geq 0$ in the proof of Theorem 18.4.

19.25 Write out a careful proof of Theorem 18.5 for mixed constraints.

CHAPTER 20

Homogeneous and Homothetic Functions

Chapters 14 and 15 examined the basic properties of differentiable functions. They showed that a lot of information can be gleaned from the fact that a differentiable function is well approximated at each point by a linear function. Economists often work with functions which have other strong properties, such as homogeneity or convexity. Sometimes, these properties arise naturally for specific functions; for example, demand functions are naturally homogeneous in prices and income. Other times, economists make these assumptions in order to prove theorems about economic models; for example, we can say a lot more about models with homothetic utility functions or concave profit functions than we can without such assumptions.

The next two chapters will examine the important properties of special kinds of functions which arise in economic models. There are two basic categories of such functions: homogeneous functions and concave/convex functions. Each of these categories has a cardinal and an ordinal component — concepts that we will develop in Section 20.4. As we will see, homogeneity and concavity are cardinal properties; *homotheticity* is the ordinal analogue of homogeneity and *quasiconcavity* is the ordinal analogue of concavity.

Each of these classes are defined without regard to the differentiability of the function. However, we can and will develop especially strong results for differentiable functions in each of these categories. In particular, we will prove simple calculus-based criteria for determining whether or not a given differentiable function is in any of these classes.

20.1 HOMOGENEOUS FUNCTIONS

Definition and Examples

Homogeneous functions arise naturally throughout economics. Profit functions and cost functions that are derived from production functions, and demand functions that are derived from utility functions are automatically homogeneous in the standard economic models.

A specific homogeneous functional form which economists frequently use as a production or utility function is the **Cobb-Douglas function**

$$q = Ax_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}, \tag{3}$$

a monomial with exponents $a_1, \ldots, a_n$ that are usually positive fractions. Since the pioneering work of mathematician C. W. Cobb and economist (and later U.S. Senator) Paul Douglas in the 1920s, economists interested in estimating the production function of a specific firm or industry will often try to find the Cobb-Douglas production function which best fits the firm's input-output data. They can often use linear ordinary least squares techniques since by taking the logarithm of both sides of function (3), they can work with the log of the output as a *linear* function of the logs of the inputs:

$$\log q = \log A + a_1 \log x_1 + \cdots + a_n \log x_n.$$

Notice that a Cobb-Douglas production function exhibits decreasing, constant, or increasing returns to scale according to whether the sum of its exponents is less than, equal to, or greater than 1. Economists have usually found in their empirical studies that this sum is very close to 1.

While production functions are often homogeneous *by assumption*, demand functions are homogeneous *by nature* (at least if we ignore the "money illusion"). Recall that a demand function $\mathbf{x} = D(p_1, \ldots, p_n, I)$ associates to each price vector $\mathbf{p} = (p_1, \ldots, p_n)$ and income level I, an individual's most-preferred consumption bundle $\mathbf{x}$ at those prices and income. It is the solution of the basic consumer maximization problem: $\mathbf{x} = D(\mathbf{p}, I)$ maximizes $U(\mathbf{x})$ subject to the constraints $x_i \geq 0$ for all i and

$$p_1x_1 + \cdots + p_nx_n \leq I. \tag{4}$$

Notice that if all the prices and the consumer's income tripled, constraint (4) would not change. We could just divide the new inequality (4) through by 3 to return to the original inequality. In particular, the optimal consumption bundle $\mathbf{x}$ would not be affected. In terms of the demand function,

$$D(tp_1, \ldots, tp_n, tI) = D(p_1, \ldots, p_n, I) \quad \text{for all } p_1, \ldots, p_n, I. \tag{5}$$

Since $t^0 = 1$, equation (5) states that demand is homogeneous of degree zero in $\mathbf{p}$ and I. Since each individual demand function is homogeneous of degree zero, the sum of these individual demands, aggregate demand, is also homogeneous of degree zero. Theorems 22.3 and 22.4 present some specific economic principles that are consequences of the homogeneity of demand functions.

Finally, a similar, straightforward calculation shows that for a firm in a competitive market, the (minimal) cost function is a homogeneous function of input prices and the optimal profit function is a homogeneous function of output price.

Properties of Homogeneous Functions

Homogeneity is a rather strong assumption for a production function and especially for a utility function. We next look at the consequences of choosing a homogeneous function by answering the following questions:

(1) What can one say about the level sets of a homogeneous function?
(2) What useful analytical properties do homogeneous functions have?

First, we prove a rather intuitive property of differentiable homogeneous functions — that the partial derivatives of a function homogeneous of degree k are themselves homogeneous of degree $k - 1$. This property is rather obvious for homogeneous *polynomials*. The following theorem proves it for general homogeneous functions.

Theorem 20.1 Let $z = f(\mathbf{x})$ be a C^1 function on an open cone in $\mathbf{R}^n$. If f is homogeneous of degree k, its first order partial derivatives are homogeneous of degree $k - 1$.

Proof For simplicity of notation, we prove this theorem for $\partial f/\partial x_1$. By hypothesis,

$$f(tx_1, tx_2, \ldots, tx_n) = t^k f(x_1, x_2, \ldots, x_n). \tag{6}$$

Think of (6) as an expression in the $n+1$ variables $t, x_1, \ldots, x_n$. Hold $t, x_2, \ldots, x_n$ fixed in expression (6) and take the partial derivative of both sides of (6) with respect to x_1. By the Chain Rule, the result is

$$\frac{\partial f}{\partial x_1}(tx_1, \ldots, tx_n) \cdot t = t^k \frac{\partial f}{\partial x_1}(x_1, \ldots, x_n),$$

or, dividing both sides by t,

$$\frac{\partial f}{\partial x_1}(t\mathbf{x}) = t^{k-1}\frac{\partial f}{\partial x_1}(\mathbf{x}). \quad \blacksquare$$

The basic geometric property of homogeneous functions is a direct consequence of the definition of homogeneous. Let $q = f(\mathbf{x})$ be a production function that is homogeneous of degree one. In Figure 20.1, we have labeled as $\mathbf{x}_i$ four points on the isoquant for $\{q = 1\}$. Let $\mathbf{w}_i = 2\mathbf{x}_i$ for $i = 1, 2, 3, 4$. Since f is homogeneous of degree one,

$$f(\mathbf{w}_i) = f(2\mathbf{x}_i) = 2f(\mathbf{x}_i) = 2.$$

The $\mathbf{w}_i$'s are all on the isoquant $\{q = 2\}$. More generally, if we translate each point $\mathbf{x}$ on the isoquant $\{q = 1\}$ by a factor r along rays from the origin, we generate the isoquant $\{q = r\}$. If f is homogeneous of degree k, then if we translate points on the isoquant $\{q = 1\}$ by a factor r along rays from the origin, we generate the isoquant $\{q = r^k\}$, since $f(r\mathbf{x}) = r^k f(\mathbf{x}) = r^k$ if $f(\mathbf{x}) = 1$. In summary, the level sets of a homogeneous function are *radial expansions and contractions* of each other.

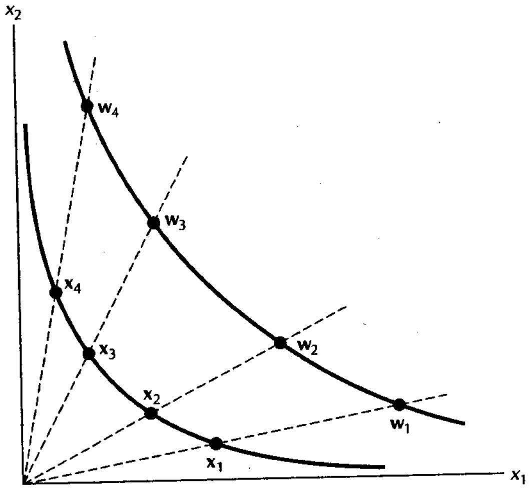

Figure 20.1 *$f(2\mathbf{x}_i) = 2f(\mathbf{x}_i) = 2$ if f is homogeneous of degree one and $f(\mathbf{x}_i) = 1$.*

One consequence of this observation is expressed in the following theorem.

Theorem 20.2 Let $q = f(\mathbf{x})$ be a C^1 homogeneous function on the positive orthant. The tangent planes to the level sets of f have constant slope along each ray from the origin.

Proof For simplicity, we will prove this theorem for a homogeneous production function on $\mathbf{R}^2_+$. Basically we want to show that the marginal rate of technical substitution (MRTS) is constant along rays from the origin. Let (L_0, K_0) and $(L_1, K_1) = t(L_0, K_0)$ be two input bundles on the same ray from the origin, as illustrated in Figure 20.2. We write f'_L for $\partial f/\partial L$. The MRTS at (L_1, K_1) equals

$$\frac{f'_L(L_1, K_1)}{f'_K(L_1, K_1)} = \frac{f'_L(tL_0, tK_0)}{f'_K(tL_0, tK_0)} \qquad \text{(by definition of } (L_1, K_1)\text{)},$$

$$= \frac{t^{k-1} f'_L(L_0, K_0)}{t^{k-1} f'_K(L_0, K_0)} \qquad \text{(by Theorem 20.1)},$$

$$= \frac{f'_L(L_0, K_0)}{f'_K(L_0, K_0)} \qquad \text{(the MRTS at } (L_0, K_0)\text{)}. \quad \blacksquare$$

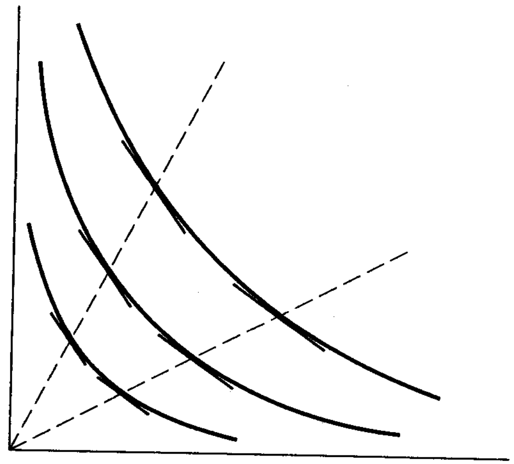

Figure 20.2 *The MRTS of a homogeneous function is constant along rays from* **0**.

Theorem 20.2 has important consequences for utility and production functions. For example, suppose that $U(\mathbf{x})$ is a homogeneous utility function. Fix prices at $\mathbf{p} = (p_1, \ldots, p_n)$ and fix income at I_0. Consider once again the problem of maximizing $U(\mathbf{x})$ subject to the budget constraint $p_1x_1 + \cdots + p_nx_n \leq I_0$. The usual geometric solution to this problem is presented in Figure 20.3. At the maximizer $\mathbf{x}(I_0)$, the level curve of U is tangent to the budget line. Analytically, at $\mathbf{x}(I_0)$ the slope of the level curve (or the marginal rate of substitution), $-U'_{x_1}/U'_{x_2}$, equals the slope of the budget line, $-p_1/p_2$.

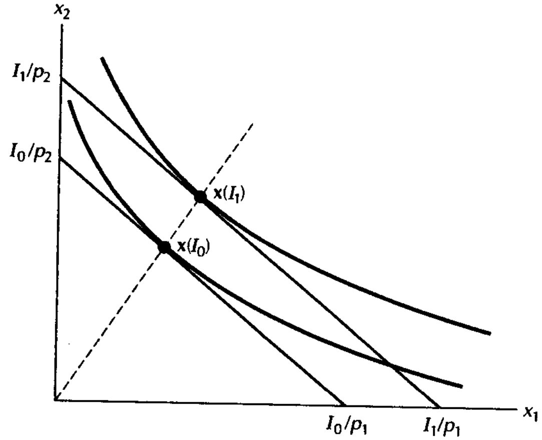

Figure 20.3 *Bundle $\mathbf{x}(I_0)$ maximizes utility on the budget set for income I_0.*

Now increase income by a factor of r to I_1, while holding prices constant. The corresponding budget line will move out *parallel* to itself, as in Figure 20.3. Its slope remains $-p_1/p_2$. The solution to the new utility maximization problem occurs at the point on the new budget line where the marginal rate of substitution equals $-p_1/p_2$. Since the utility function is homogeneous, this point will lie at the intersection of the new budget line and the ray from the origin through $\mathbf{x}(I_0)$, as in Figure 20.3, by Theorem 20.2. The parameterized curve $I \mapsto \mathbf{x}(I)$ in Figure 20.3 that indicates the bundle demanded for different income levels is called the **income expansion path**. We have just shown that the income expansion path for a *homogeneous* utility function is a ray from the origin.

Since the budget line in Figure 20.3 moved out by a factor r, the new bundle of choice $\mathbf{x}(I_1)$ is a multiple of the former one by a factor r. Analytically, $\mathbf{x}(I_1) = \mathbf{x}(rI_0) = r\mathbf{x}(I_0)$. In other words, for a homogeneous utility function of degree k, the corresponding demand function is a homogeneous function of degree one in income; doubling income doubles consumption of every good.

The fact that demand as a function of the single variable income is homogeneous of degree one in this model means that every component $x_i(I)$ of $\mathbf{x}(I)$ is a linear function of income: $x_i(I) = a_iI$, by Example 20.4. It follows that each *income elasticity of demand* is identically 1, since $x_i = a_iI$ implies

$$\frac{dx_i}{dI} \cdot \frac{I}{x_i} = a_i \cdot \frac{I}{a_iI} = 1.$$

Given a production function $q = f(\mathbf{x})$ and a cost C of inputs, the firm wants to choose the input bundle $\mathbf{x}$ that maximizes revenue $pf(\mathbf{x})$, subject to $\mathbf{w} \cdot \mathbf{x} \leq C$. If the production function is homogeneous, the above analysis shows that the optimal choice of each input is a linear function of cost: $x_i(C) = a_iC$. Plugging these expressions into the homogeneous production function yields

$$\begin{aligned} q = q(C) &= f(x_1(C), \ldots, x_n(C)) \\ &= f(a_1C, \ldots, a_nC) = C^k f(a_1, \ldots, a_n) \\ &= C^k a^*. \end{aligned}$$

Therefore, the cost function — the function that relates input cost and optimal output — is $C(q) = bq^{1/k}$, where $b = (a^*)^{-1/k}$. We summarize the results of this discussion in the following theorem.

Theorem 20.3 Let $U(\mathbf{x})$ be a utility function on $\mathbf{R}^n_+$ that is homogeneous of degree k. Then,

(*i*) the MRS is constant along rays from the origin,
(*ii*) income expansion paths are rays from the origin,
(*iii*) the corresponding demand depends linearly on income, and
(*iv*) the income elasticity of demand is identically 1.

Let $q = f(\mathbf{x})$ be a production function on $\mathbf{R}^n_+$ that is homogeneous of degree k. Then,

(*i*) the marginal rate of technical substitution (MRTS) is constant along rays from the origin, and
(*ii*) the corresponding cost function is homogeneous of degree $1/k$: $C(q) = bq^{1/k}$.

A Calculus Criterion for Homogeneity

We complete our discussion of homogeneous functions by presenting a calculus criterion which is a necessary and sufficient condition for a C^1 function to be homogeneous. The necessary condition, commonly known as Euler's theorem, is a useful analytic tool in working with homogeneous functions. This condition is related to the fact that when you take the derivative of a monomial, you multiply its coefficient by the original exponent and then lower the exponent by 1: $(ax^k)' = kax^{k-1}$. Therefore,

$$x(ax^k)' = k(ax^k); \quad \text{that is,} \quad xf'(x) = kf(x).$$

The following theorem is the n-dimensional version of this result.

Theorem 20.4 **(Euler's theorem)** Let $f(\mathbf{x})$ be a C^1 homogeneous function of degree k on $\mathbf{R}^n_+$. Then, for all $\mathbf{x}$,

$$x_1\frac{\partial f}{\partial x_1}(\mathbf{x}) + x_2\frac{\partial f}{\partial x_2}(\mathbf{x}) + \cdots + x_n\frac{\partial f}{\partial x_n}(\mathbf{x}) = kf(\mathbf{x}), \tag{7}$$

or, in gradient notation,

$$\mathbf{x} \cdot \nabla f(\mathbf{x}) = kf(\mathbf{x}).$$

Proof Simply differentiate each side of the definition (1) of homogeneous function with respect to t and then set $t = 1$:

$$\frac{d}{dt}f(tx_1, \ldots, tx_n) = \frac{\partial f}{\partial x_1}(t\mathbf{x})x_1 + \cdots + \frac{\partial f}{\partial x_n}(t\mathbf{x})x_n$$

$$\frac{d}{dt}[t^k f(x_1, \ldots, x_n)] = kt^{k-1}f(x_1, \ldots, x_n).$$

The two left-hand sides are equal by the definition of homogeneous. Set $t = 1$ in the two right-hand sides to get the desired result (7). ∎

Though it is less frequently used, we present the converse of Euler's theorem for the sake of completeness. Its proof involves the use of differential equations and will be presented in the Appendix of Chapter 24.

Theorem 20.5 Suppose that $f(x_1, \ldots, x_n)$ is a C^1 function on the positive orthant $\mathbf{R}^n_+$. Suppose that

$$x_1 \frac{\partial f}{\partial x_1}(\mathbf{x}) + \cdots + x_n \frac{\partial f}{\partial x_n}(\mathbf{x}) = kf(x_1, \ldots, x_n)$$

for all $\mathbf{x}$ in $\mathbf{R}^n_+$. Then, f is homogeneous of degree k.

Economic Applications of Euler's Theorem

A standard application of Euler's Theorem in economics is the story of "product exhaustion" for firms with homogeneous production functions. If a firm has a production function $q = f(x_1, \ldots, x_n)$ that is homogeneous of degree one, then (7) becomes

$$x_1 \frac{\partial f}{\partial x_1}(\mathbf{x}) + \cdots + x_n \frac{\partial f}{\partial x_n}(\mathbf{x}) = f(\mathbf{x}) = q. \tag{8}$$

For each input, multiply the amount used, x_i, by its marginal product $\partial f/\partial x_i$, and sum over all the inputs. The result, according to (8), is the amount of output q. To understand the implication of (8), suppose the usual profit-maximizing criterion, namely that the firm pays each factor x_i its marginal revenue product $p \cdot (\partial f/\partial x_i)$, so that it hires each factor until the contribution of that factor to the output of the firm just equals the cost of acquiring additional units of that factor. (See Section 17.5.) Then, the firm's total payment will be

$$x_1 p \frac{\partial f}{\partial x_1}(\mathbf{x}) + \cdots + x_n p \frac{\partial f}{\partial x_n}(\mathbf{x}).$$

But by equation (8), this is just $p \cdot q$, the value of the firm's output. So the revenue of the firm with a constant-returns-to-scale production function is exactly exhausted in making payments to all the factors. Such firms make zero economic profit. If the degree of homogeneity were greater than one, total payments would exceed the value of output; if the degree were less than one, total payments would be less than the value of output and the firm would make a positive profit.

As another application of Euler's theorem, let $q = f(x_1, x_2)$ be a production function which satisfies:

(1) constant returns to scale: $f(t\mathbf{x}) = tf(\mathbf{x})$, and
(2) decreasing marginal product of x_1: $\partial^2 f/\partial x_1^2 < 0$.

Since f is homogeneous of degree one, its partial derivative $\partial f/\partial x_1$ is homogeneous of degree zero. Apply Euler's theorem to $\partial f/\partial x_1$:

$$0 \cdot \frac{\partial f}{\partial x_1} = x_1 \cdot \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_1}\right) + x_2 \cdot \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_1}\right),$$

or

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = -\frac{x_1}{x_2}\frac{\partial^2 f}{\partial x_1^2},$$

which is positive since $f''_{x_1x_1} < 0$. This positive cross partial derivative means that the marginal product of one factor increases when the other factor is increased. This result is sometimes called **Wicksell's law**.

EXERCISES

20.1 Which of the following functions are homogeneous? What are the degrees of homogeneity of the homogeneous ones?

a) $3x^5y + 2x^2y^4 - 3x^3y^3$, *b)* $3x^5y + 2x^2y^4 - 3x^3y^4$,
c) $x^{1/2}y^{-1/2} + 3xy^{-1} + 7$, *d)* $x^{3/4}y^{1/4} + 6x$,
e) $x^{3/4}y^{1/4} + 6x + 4$, *f)* $\dfrac{(x^2 - y^2)}{(x^2 + y^2)} + 3$.

20.2 Verify Euler's theorem for the functions in Examples 20.1 and 20.3.

20.3 Prove that the product of homogeneous functions is homogeneous.

20.4 Consider the constant elasticity of substitution (CES) production function $F(x_1, x_2) = A\left(a_0 + a_1x_1^\rho + a_2x_2^\rho\right)^{1/\rho}$. Show that F has constant returns to scale when $a_0 = 0$.

20.5 If $y = f(x_1, x_2)$ is C^2 and homogeneous of degree r, show that

$$x_1^2 f''_{x_1x_1} + 2x_1x_2 f''_{x_1x_2} + x_2^2 f''_{x_2x_2} = r(r-1)f.$$

20.6 Prove that if f and g are functions on $\mathbf{R}^n$ that are homogeneous of *different* degrees, then $f + g$ is not homogeneous.

20.7 Is the zero function $f(\mathbf{x}) \equiv 0$ homogeneous? If so, of what degree? How does your answer relate to the previous exercise?

20.2 HOMOGENIZING A FUNCTION

Homogeneous functions have so many nice properties and arise so naturally in applications that it is natural to ask whether any arbitrary function can be considered as the restriction of a homogeneous function that is defined on a higher-dimensional space. The answer to this question is a definite yes, and the construction is fairly straightforward.

Theorem 20.6 Let $(x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n)$ be a real-valued function defined on a cone C in $\mathbf{R}^n$. Let k be an integer. Define a new function F of $n+1$ variables by

$$F(x_1, \ldots, x_n, z) = z^k \cdot f\left(\frac{x_1}{z}, \ldots, \frac{x_n}{z}\right). \tag{9}$$

Then, F is a homogeneous function of degree k on the cone $C \times \mathbf{R}_+$ in $\mathbf{R}^{n+1}$. Since $f(\mathbf{x}) = F(\mathbf{x}, 1)$ for all $\mathbf{x} \in C$, we can consider f as the restriction of F to an n-dimensional subset of $\mathbf{R}^{n+1}$.

Proof For any $t \in \mathbf{R}_+$ and $(\mathbf{x}, z) \in C \times \mathbf{R}_+$,

$$\begin{aligned} F(t\mathbf{x}, tz) &= (tz)^k f\left(\frac{1}{tz} t\mathbf{x}\right) && \text{(by the definition (9) of } F\text{)} \\ &= t^k \cdot z^k f\left(\frac{1}{z}\mathbf{x}\right) \\ &= t^k F(\mathbf{x}, z) && \text{(by the definition (9) of } F\text{).} \quad \blacksquare \end{aligned}$$

The converse of Theorem 20.6 is also true. If F is a homogeneous extension of f, then F and f must be related by (9).

Theorem 20.7 Suppose that $(\mathbf{x}, z) \mapsto F(\mathbf{x}, z)$ is a function that is homogeneous of degree k on a set $C \times \mathbf{R}_+$ for some cone C in $\mathbf{R}^n$ and that

$$F(\mathbf{x}, 1) = f(\mathbf{x}) \quad \text{for all } \mathbf{x} \in C. \tag{10}$$

Then, $F(\mathbf{x}, z) = z^k f\left(\frac{1}{z}\mathbf{x}\right)$ for all $(\mathbf{x}, z) \in C \times \mathbf{R}_+$.

Proof Since F is homogeneous of degree k,

$$\begin{aligned} F(\mathbf{x}, z) &= F\left(z \cdot \left(\frac{1}{z}\mathbf{x}, 1\right)\right) \\ &= z^k \cdot F\left(\frac{1}{z}\mathbf{x}, 1\right) \\ &= z^k \cdot f\left(\frac{1}{z}\mathbf{x}\right) && \text{(by (10)).} \quad \blacksquare \end{aligned}$$

Example 20.5 If $f(x) = x^a$ on $\mathbf{R}_+$, then its homogenization of degree one is

$$F(x, y) = y \cdot \left(\frac{x}{y}\right)^a = x^a y^{1-a}.$$

Example 20.6 If f is the nonhomogeneous function $x \mapsto x - ax^2$, then its degree-one homogenization is

$$\begin{aligned} F(x, y) &= y \cdot f\left(\frac{x}{y}\right) \\ &= y\left[\left(\frac{x}{y}\right) - a\left(\frac{x}{y}\right)^2\right] \\ &= x - a\frac{x^2}{y}. \end{aligned}$$

Economic Applications of Homogenization

If we are given a function f of $n-1$ variables and we know that it is the restriction of some homogeneous function F of n variables, we can use Theorems 20.6 and 20.7 to construct F from f. For example, f might be a production function that has been estimated using an incomplete list of factors $x_1, \ldots, x_{n-1}$. Suppose that there is one unestimated factor and that the complete production function of all n factors is known to have constant returns to scale. By Theorem 20.7, $F(x_1, \ldots, x_n) = x_n \cdot f\left(\frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}\right)$. With this explicit formula for F, one can compute such things as the marginal product of the hidden factor.

Example 20.7 In a two-factor constant-returns-to-scale production process, an econometrician estimates that when the second factor is held constant, the production function for the first factor is $f_1(x_1) = x_1^a$ for some $a \in (0, 1)$. Then, the complete production function would be the Cobb-Douglas production function $F(x_1, x_2) = x_1^a x_2^{1-a}$, as we computed in Example 20.5. If units are chosen so that $x_2 = 1$ during the estimation of f_1, then the estimated function is the restriction $f_1(x_1) = F(x_1, 1)$. The marginal product of the hidden factor x_2 when $x_2 = 1$ is

$$\begin{aligned} \frac{\partial F}{\partial x_2}(x_1, 1) &= (1-a)x_1^a \cdot x_2^{-a}\Big|_{x_2=1} \\ &= (1-a)f(x_1) \end{aligned}$$

in the specially chosen units of x_2 for which $f(x_1) = F(x_1, 1)$.

In consumer theory, we know that demand functions must be homogeneous of degree zero in all commodities. Suppose, for example, that we are studying a two-good market, say cookies and milk. Suppose that we calculate the demand $D_1(p_1)$ for milk in a situation where the price of cookies is held constant. To obtain the demand function for milk as a function of both prices, we simply homogenize the milk demand function, using (9) with $k = 0$:

$$D(p_1, p_2) = D_1\left(\frac{p_1}{p_2}\right)$$

in units such that $p_2 = 1$ in the estimation of D_1.

Example 20.8 For example, if the demand function for milk with p_2 held constant at $p_2 = 1$ is the constant elasticity function $Q_1 = bp_1^{-a}$, then the complete demand function for milk is $Q_1 = bp_1^{-a}p_2^a$, a homogeneous function of degree zero, as it should be.

EXERCISES

20.8 Write the degree-one homogenization of each of the following functions:

$a)\ e^x, \quad b)\ \ln x, \quad c)\ 5, \quad d)\ x_1^2 + x_2^3, \quad e)\ x_1^2 + x_2^2.$

20.3 CARDINAL VERSUS ORDINAL UTILITY

As Theorem 20.3 indicates, homogeneous functions have some properties that make them useful functional forms for utility or production functions. But modern utility is an *ordinal* theory, not *cardinal*. And homogeneity is a *cardinal* property, not *ordinal*. This section will clarify the meaning of the concepts *cardinal* and *ordinal*, and will look at the ordinal content of homogeneity. The next section will look at the larger class of all functions which have these same ordinal properties. They are called *homothetic functions*.

A utility function could be said to measure the level of satisfaction associated with each commodity bundle. However, no economist really believes that a real number can be assigned to each commodity bundle which expresses (in utils?) the consumer's level of satisfaction with that bundle. Economists do believe that consumers have well-behaved preferences over bundles and that, given any two bundles, a consumer can indicate a preference of one over the other or indifference between the two. Although economists usually work with utility functions, they are really only concerned with the level sets of such functions, not with the number which the utility function assigns to any given level set. In utility theory, these level sets are called indifference sets, or indifference curves when the level sets are curves. A property of utility functions is called **ordinal** if it depends only on the shape and location of a consumer's indifference sets. On the other hand, a property is called **cardinal** if it also depends on the actual amount of utility that the utility function assigns to each indifference set.

In this context, we say two functions are **equivalent** if they have the exact same indifference sets, although they may assign different numbers to any given indifference set. For example, let $u(x, y)$ be a utility function on $\mathbf{R}^2_+$. Let $v(x, y)$ be the utility function $u(x, y) + 1$. These two functions have the exact same set of indifference curves. The function v assigns a number one unit larger than the number that the function u assigns to each indifference curve. For example, the indifference curve $\{u = 13\}$ coincides with the indifference curve $\{v = 14\}$. The functions u and v represent the same preferences and are therefore equivalent. As a second example, the utility function $w(x, y) = [u(x, y)]^2$ is also equivalent to u. If

$$w(x_1, y_1) = w(x_2, y_2) = a, \quad \text{then} \quad u(x_1, y_1) = u(x_2, y_2) = \sqrt{a}.$$

To all bundles which w assigns utility 9, u assigns utility 3, and vice versa. The utility functions u and w have the same indifference curves; they just attach different numbers to them. If $g_1(z) = z + 1$ and $g_2(z) = z^2$, then we can write $v = g_1 \circ u$ and $w = g_2 \circ u$. We say that v and w are monotonic transformations of u.

Definition Let I be an interval on the real line. Then, $g : I \to \mathbf{R}$ is a **monotonic transformation** of I if g is a strictly increasing function on I. Furthermore, if g is a monotonic transformation and u is a real-valued function of n variables, then we say that

$$g \circ u : \mathbf{x} \mapsto g(u(\mathbf{x}))$$

is a **monotonic transformation of** u.

Of course, if g is differentiable, then g is a monotonic transformation if $g'(x) > 0$ for all x in I. (We could allow such a g to have a zero derivative at isolated points. For example, z^3 is strictly increasing, even though its derivative is zero at $z = 0$ and positive everywhere else.)

Example 20.9 The functions

$$3z + 2, \quad z^2, \quad e^z, \quad \text{and} \quad \ln z$$

are all monotonic transformations of $\mathbf{R}_{++}$, the set of all positive scalars. Consequently, the utility functions

$$3xy + 2, \quad (xy)^2, \quad (xy)^3 + xy, \quad e^{(xy)}, \quad \text{and} \quad \ln xy = \ln x + \ln y \tag{11}$$

are monotonic transformations of the utility function $u(x, y) = xy$.

We can now give a precise definition of an ordinal property.

Definition A characteristic of functions is called **ordinal** if every monotonic transformation of a function with this characteristic still has this characteristic.

Cardinal properties are not preserved by monotonic transformations.

Example 20.10 Consider the class of utility functions on $\mathbf{R}^2_+$ that are monomials — polynomials with only one term; for example, the polynomial $u(x, y) = x^2y$. The utility function $v(x, y) = x^2y + 1$ is a monotonic transformation of u. As we discussed above, both u and v have the same indifference curves. However, v is not a monomial. So, being monomial is a cardinal property. We should be uncomfortable with any theorem which only holds for *monomial* utility functions.

Example 20.11 A utility function $u(x_1, x_2)$ is **monotone in** x_1 if for each fixed x_2, u is an increasing function of x_1. If u is differentiable, we could write this property as $\partial u/\partial x_1 > 0$. Intuitively, monotonicity in x_1 means that increasing consumption of commodity one increases utility; in other words, commodity one is a *good*. This property depends only on the shape and location of the level sets of u and on the direction of higher utility. Therefore, it is an ordinal property. Analytically, if $g(z)$ is a monotonic transformation with $g' > 0$, then by the Chain Rule

$$\frac{\partial}{\partial x_1}[g(u(x_1, x_2)] = g'(u(x_1, x_2)) \cdot \frac{\partial u}{\partial x_1}(x_1, x_2) > 0.$$

Example 20.12 Because of their preference for ordinal concepts over cardinal concepts, economists would much rather work with the marginal rate of substitution (MRS) than with the marginal utility (MU) of any given utility function, because MU is a cardinal concept. For example, if $v = 2u$,

$$\frac{\partial v}{\partial x_1}(x_1^*, x_2^*) = 2\frac{\partial u}{\partial x_1}(x_1^*, x_2^*).$$

Thus, equivalent utility functions have different marginal utilities at the same bundle. On the other hand, MRS is an ordinal concept. Let v be a general monotonic transformation of u: $v(x, y) = g(u(x, y))$. The MRS for v at

$$\frac{\frac{\partial v}{\partial x}(x^*, y^*)}{\frac{\partial v}{\partial y}(x^*, y^*)} = \frac{\frac{\partial}{\partial x}g(u(x^*, y^*))}{\frac{\partial}{\partial y}g(u(x^*, y^*))}$$

$$= \frac{g'(u(x^*, y^*)) \cdot \frac{\partial u}{\partial x}(x^*, y^*)}{g'(u(x^*, y^*)) \cdot \frac{\partial u}{\partial y}(x^*, y^*)}$$

$$= \frac{\frac{\partial u}{\partial x}(x^*, y^*)}{\frac{\partial u}{\partial y}(x^*, y^*)},$$

the MRS for u at (x^*, y^*).

Remark In dealing with production functions, we care a lot about the number that a production function assigns to any isoquant. The level of output for each input has full meaning here. In other words, the distinction between cardinal and ordinal is of no concern when we are speaking about production functions.

EXERCISES

20.9 For each of the five utility functions in (11) in Example 20.9, identify the level sets which correspond to the level sets $\{xy = 1\}$ and $\{xy = 4\}$ of u. For example, the level set $\{xy = 1\}$ corresponds to the level set $\{3xy + 2 = 5\}$. In each case, convince yourself that these level curves are indeed identical by finding four bundles on the level set of xy and showing that these bundles are on the corresponding level sets of the other five utility functions.

20.10 Show directly that each of the five equivalent utility functions in Example 20.9 have the same marginal rates of substitution at the bundle (2, 1). Show that they have different marginal utilities (of good one) at (2, 1).

20.11 Which of the following are monotonic transformations of $\mathbf{R}_+$?

a) $z^4 + z^2$, *b)* $z^4 - z^2$, *c)* $z/(z + 1)$, *d)* $\sqrt{z}$, *e)* $\sqrt{z^2 + 4}$.

20.12 Which of the following functions are equivalent to xy? For those which are, what monotonic transformation provides this equivalence?

a) $7x^2y^2 + 2$, *b)* $\ln x + \ln y + 1$, *c)* x^2y, *d)* $x^{1/3}y^{1/3}$.

20.13 Use the monotonic transformation z^k to prove that every homogeneous function is equivalent to a homogeneous function of degree one.

20.14 Is having decreasing marginal utility, $(\partial^2 U/\partial x_i^2) < 0$ for all i, an ordinal property? Why?

20.15 Prove that any function $f: \mathbf{R}^1 \to \mathbf{R}^1$ with $f' > 0$ everywhere is equivalent to a homogeneous function of degree one.

20.4 HOMOTHETIC FUNCTIONS

Motivation and Definition

As we stated at the beginning of the previous section, homogeneity is a cardinal property, not an ordinal one. We need only one example to verify this, but we will present two. The functions $g_1(z) = z^3 + z$ and $g_2(z) = z + 1$ are both monotonic transformations. However, if we apply these transformations to the homogeneous function $u(x, y) = xy$, we obtain the *nonhomogeneous functions* $v(x, y) = x^3y^3 + xy$ and $w(x, y) = xy + 1$.

Nevertheless, as Theorem 20.3 indicates, many of the important properties that make homogeneous functions so useful in utility theory are ordinal properties:

(1) Level sets are radial expansions and contractions of each other.
(2) The slope of level sets is constant along rays from the origin.

These two properties are clearly ordinal; they pertain only to the shape and slopes of level curves with no concern at all about the numbers attached to these level sets. Their consequences for demand theory are described in Theorem 20.3: Income expansion paths are rays coming out of the origin, and the income elasticity of demand is everywhere 1.

We now define a class of ordinal functions—a class that has all the ordinal properties that homogeneous functions have.

Definition A function $v: \mathbf{R}^n_+ \to \mathbf{R}$ is called **homothetic** if it is a monotone transformation of a homogeneous function, that is, if there is a monotonic transformation $z \mapsto g(z)$ of $\mathbf{R}_+$ and a homogeneous function $u : \mathbf{R}^n_+ \to \mathbf{R}_+$ such that $v(\mathbf{x}) = g(u(\mathbf{x}))$ for all $\mathbf{x}$ in the domain.

Example 20.13 The two functions at the beginning of this section,

$$v(x, y) = x^3y^3 + xy \quad \text{and} \quad w(x, y) = xy + 1,$$

are homothetic functions with $u(x, y) = xy$ and with $g_1(z) = z^3 + z$ and $g_2(z) = z + 1$, respectively. The five examples in Example 20.1 are homothetic functions.

It should be clear by its definition that homotheticity is an ordinal property. To prove this analytically, we need to prove that a monotonic transformation of a homothetic function is still homothetic. Let $z \mapsto h(z)$ be a monotonic transformation and let $\mathbf{x} \mapsto v(\mathbf{x})$ be a homothetic transformation. We need to check that $h \circ v$ is homothetic. By the definition of homothetic, $v(\mathbf{x})$ can be written as $v(\mathbf{x}) = g(u(\mathbf{x}))$, where g is a monotonic transformation and $z = u(\mathbf{x})$ is a homogeneous function. Now

$$h(v(\mathbf{x})) = h(g(u(\mathbf{x})) = (h \circ g)(u(\mathbf{x})).$$

Since u is homogeneous, we need only show that $h \circ g$ is a monotonic transformation, in other words, that a monotonic transformation of a monotonic transformation is still a monotonic transformation.

Let $z_2 > z_1$. Since g is strictly increasing, $g(z_2) > g(z_1)$. Since h is strictly increasing, $h(g(z_2)) > h(g(z_1))$; that is, $(h \circ g)(z_2) > (h \circ g)(z_1)$. This implies that $h \circ g$ is a monotonic transformation, and therefore that $h \circ v = (h \circ g) \circ u$ is a monotonic transformation of the homogeneous function u; that is, $h \circ v$ is homothetic.

Characterizing Homothetic Functions

This section began with a discussion of the two primary ordinal properties of homogeneous functions. As we will now see, these properties characterize homothetic utility functions. The key property is the first: level sets are radial expansions and contractions of one another. Before proving that this property characterizes homothetic utility functions, we need some definitions that extend the notion of a monotone function to higher dimensions.

Definition If $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$, write

$$\mathbf{x} \geq \mathbf{y} \quad \text{if} \quad x_i \geq y_i \text{ for } i = 1, \dots, n,$$

$$\mathbf{x} > \mathbf{y} \quad \text{if} \quad x_i > y_i \text{ for } i = 1, \dots, n,$$

A function $u: \mathbf{R}^n_+ \to \mathbf{R}$ is **monotone** if for all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n_+$,

$$\mathbf{x} \geq \mathbf{y} \implies u(\mathbf{x}) \geq u(\mathbf{y}).$$

The function u is **strictly monotone** if for all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n_+$,

$$\mathbf{x} > \mathbf{y} \implies u(\mathbf{x}) > u(\mathbf{y}).$$

Monotonicity and strict monotonicity are natural properties of utility functions in that they capture the essence of the "more is better" aspect of preferences. The following theorem gives us the promised characterization of homothetic functions.

Theorem 20.8 Let $u : \mathbf{R}^n_+ \to \mathbf{R}$ be a strictly monotone function. Then, u is homothetic if and only if for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{R}^n_+$,

$$u(\mathbf{x}) \geq u(\mathbf{y}) \iff u(\alpha\mathbf{x}) \geq u(\alpha\mathbf{y}) \quad \text{for all } \alpha > 0. \tag{12}$$

Proof We first show that if u satisfies (12), it is homothetic. Let $\mathbf{e}$ denote the vector $(1, 1, \ldots, 1)$, that spans the diagonal Δ in $\mathbf{R}^n$. Define function $f: \mathbf{R}_+ \to \mathbf{R}$ by

$$f(t) = u(t\mathbf{e}).$$

Since u is strictly increasing, so is f; and therefore, f has a strictly increasing inverse g. Let $v = g \circ u$. Then,

$$f \circ v = f \circ (g \circ u) = (f \circ g) \circ u = u.$$

To prove that $u = f \circ v$ is homothetic, we need only show that v is homogeneous.

For any scalar a, the function $a \longmapsto g(a)$ tells how far up the diagonal Δ the level set $u^{-1}(a)$ meets Δ. Consequently, $v(\mathbf{x}) = g(u(\mathbf{x}))$ tells how far up Δ the u-level set through $\mathbf{x}$ crosses Δ. Analytically, $t = v(\mathbf{x})$ is the solution of

$$u(\mathbf{x}) = u(t\mathbf{e}). \tag{13}$$

Let $\alpha > 0$ be a scalar. By (12) and Exercise 20.20,

$$u(\mathbf{x}) = u(t\mathbf{e}) \implies u(\alpha\mathbf{x}) = u(\alpha t\mathbf{e}). \tag{14}$$

But, (14) indicates that αt is the solution of (13) with $\alpha\mathbf{x}$ replacing $\mathbf{x}$. In other words, $v(\alpha\mathbf{x}) = \alpha v(\mathbf{x})$; v is homogeneous of degree one. Since v is homogeneous and f is increasing, $u = f \circ v$ is homothetic.

To prove the converse, suppose first that u is *linear homogeneous*, that is, homogeneous of degree 1, and that $u(\mathbf{x}) \geq u(\mathbf{y})$ and $\alpha > 0$. These two properties yield

$$\begin{aligned} u(\alpha\mathbf{x}) &= \alpha u(\mathbf{x}) \\ &\geq \alpha u(\mathbf{y}) \\ &= u(\alpha\mathbf{y}); \end{aligned}$$

so, property (12) holds.

More generally, suppose that u is homothetic, so that $u = g_1 \circ v$, with g_1 increasing and v homogeneous of degree k. Write v as $g_2 \circ h$, where $g_2(z) = z^k$ and $h(\mathbf{x}) = v(\mathbf{x})^{1/k}$. One checks easily that v is homogeneous of degree one and that g_2 is increasing, so that we can write u as $u = f \circ h$ with $f \equiv g_1 \circ g_2$ increasing and h linear homogeneous.

Once again, suppose $u(\mathbf{x}) \geq u(\mathbf{y})$ and $\alpha > 0$. Since f is strictly increasing, it has a strictly increasing inverse f^{-1}:

$$\begin{aligned} f^{-1}(u(\mathbf{x})) &\geq f^{-1}(u(\mathbf{y})), \\ v(\mathbf{x}) &\geq v(\mathbf{y}), \\ v(\alpha\mathbf{x}) = \alpha v(\mathbf{x}) &\geq \alpha v(\mathbf{y}) = v(\alpha\mathbf{y}), \\ f(v(\alpha\mathbf{x})) &\geq f(v(\alpha\mathbf{y})), \\ u(\alpha\mathbf{x}) &\geq u(\alpha\mathbf{y}); \end{aligned}$$

and so u satisfies property (12). ■

The second ordinal property of homogeneity is that the slope of level sets is constant along rays from the origin. This property provides a calculus-based necessary condition for homotheticity, just as Euler's theorem does for homogeneity.

Theorem 20.9 Let u be a C^1 function on $\mathbf{R}^n_+$. If u is homothetic, then the slopes of the tangent planes to the level sets of u are constant along rays from the origin; in other words, for *every* i, j and for every $\mathbf{x}$ in $\mathbf{R}^n_+$,

$$\frac{\dfrac{\partial u}{\partial x_i}(t\mathbf{x})}{\dfrac{\partial u}{\partial x_j}(t\mathbf{x})} = \frac{\dfrac{\partial u}{\partial x_i}(\mathbf{x})}{\dfrac{\partial u}{\partial x_j}(\mathbf{x})} \quad \text{for all } t > 0. \tag{15}$$

Theorem 20.9 states that if u is homothetic, then its marginal rate of substitution is a homogeneous function of degree zero.

Proof The proof is a straightforward combination of the proofs of Theorem 20.2 and Example 20.12, and will be left as an exercise.

In fact, the converse of Theorem 20.9 is also true. It provides us with a calculus-based *sufficient* condition for showing that a given function is homothetic. Some texts *define* a function to be homothetic if its marginal rate of substitution is homogeneous of degree zero. As in the case of the converse to Euler's theorem, the proof of the converse to Theorem 20.9, which we omit, involves differential equations.

Theorem 20.10 Let u be a C^1 function on $\mathbf{R}^n_+$. If condition (15) holds for all $\mathbf{x}$ in $\mathbf{R}^n_+$, all $t > 0$, and all i, j, then u is homothetic.

EXERCISES

20.16 Using the arguments in Example 20.13 and in Exercise 20.13, show that we can replace "homogeneous" by "homogeneous of degree one" in the definition of homothetic.

20.17 Which of the following functions are homothetic? Give a reason for each answer.

a) $e^{x^2y}e^{xy^2}$, *b)* $2\log x + 3\log y$, *c)* $x^3y^6 + 3x^2y^4 + 6xy^2 + 9$,
d) $x^2y + xy$, *e)* $x^2y^2/(xy+1)$.

20.18 Use Theorems 20.9 and 20.10 to check the homotheticity of the functions in Exercise 20.17 and to determine whether or not $f(x, y) = x^4 + x^2y^2 + y^4 - 3x - 8y$ is homothetic.

20.19 Write out a complete, careful proof of Theorem 20.9.

20.20 Show that for a strictly monotone function u, the two inequalities in condition (12) can be replaced without loss of generality by equalities.

CHAPTER 21

Concave and Quasiconcave Functions

Concave functions play a role in economic theory similar to the role that homogeneous functions play. Both classes arise naturally in economic models — homogeneous functions as demand functions, concave functions as expenditure functions. Profit functions and cost functions are naturally both homogeneous and concave. Both classes have desirable properties for utility and production functions. Both classes have straightforward calculus-based characterizations — homogeneous functions via Euler's theorem, concave functions via a second derivative test. Finally, both classes are cardinal and need to be modified for full use in utility theory.

On the other hand, concavity is a concept that is very different from homogeneity. As we will see, there are functions which are homogeneous but not concave or convex, and there are functions which are concave or convex but not homogeneous. In a sense, these two properties are complementary; economists often prefer to work with production functions that have both properties.

21.1 CONCAVE AND CONVEX FUNCTIONS

Students first meet concave and convex functions in their study of functions of one variable in Calculus I, as we did in Section 3.2. The definitions of concavity and convexity are the same for functions of n variables as they are for functions of one variable.

Definition A real-valued function f defined on a convex subset U of $\mathbf{R}^n$ is **concave** if for all $\mathbf{x}, \mathbf{y}$ in U and for all t between 0 and 1,

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \geq tf(\mathbf{x}) + (1-t)f(\mathbf{y}). \quad (1)$$

A real-valued function g defined on a convex subset U of $\mathbf{R}^n$ is **convex** if for all $\mathbf{x}, \mathbf{y}$ in U and for all t between 0 and 1,

$$g(t\mathbf{x} + (1-t)\mathbf{y}) \leq tg(\mathbf{x}) + (1-t)g(\mathbf{y}). \quad (2)$$